Math 261B 8/27/2020

Geometric groups:
- topological group
- Lie / $\mathbb{R}$
- Lie / $\mathbb{C}$
- Algebraic groups / K ← alg. closed field.

$G \times G \xrightarrow{\mu} G$ $\mu(g,h) = gh$

$G \xrightarrow{i} G$ $i(g) = g^{-1}$

$\cdot \xrightarrow{e} G$

General structure of Lie Algebras

$G \times G \xrightarrow{\mu} G$

$\mathfrak{r} \subset \mathfrak{g}$

Radical = max solvable ideal

$\mathfrak{g}/\mathfrak{r}$ is semi-simple

$\oplus$ of simple Lie algs, classified by Dynkin diagrams

Lie algebras $\longleftrightarrow$ simply connected Lie groups

$\longleftrightarrow$ $Sl_n(\mathbb{C})$, $SO_n(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$, ... + exceptional

Ado's Thm: every $\mathfrak{g} \subseteq \mathfrak{gl}_n$

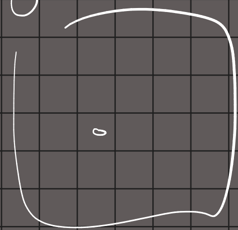
$\mathfrak{gl}_n$ = $n \times n$ matrices,

$[A, B] = AB - BA$

Simply connected $G_{/\mathbb{C}} \hookrightarrow GL_n \curvearrowright \mathbb{C}^n$

Simply connex. $G \in \mathrm{Lie}/\mathbb{C}$ are linear. $\mathbb{C}\setminus 0$

$SL_n(\mathbb{C})$ is simply connex. $SL_2(\mathbb{R})$ isn't "



$\curvearrowright \mathbb{R}^2 \setminus 0$

Simply connected cover $\widetilde{SL_2}(\mathbb{R}) \curvearrowright (\mathbb{R}^2 \setminus 0)^\sim$

$= \{(x, y, \theta)\}$

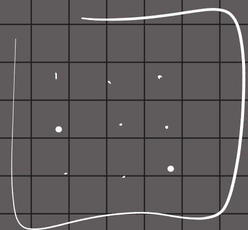
θ = a value of $\arg(x+iy)$.

$\boxed{\mathbb{C}} \xrightarrow{\exp} \mathbb{C}^\times$

is not linear.

$E = (\mathbb{C}, +)/$ lattice

(= 2-dim $\mathbb{R}$ torus.)



not linear as Lie$/\mathbb{C}$

$\mathbb{C}^n /$ rank $2n$ lattice

if nice, projective alg. variety

Abelian variety.

E is an alg. variety $/\mathbb{C}$, projective

$$E \xrightarrow[\text{alg.}]{\varphi} GL_n(\mathbb{C})$$

(connected)

coordinates x_{ij} = matrix entries

φ is constant b/c E is projective, but GL_n is affine.

Coordinates on GL_n $\qquad$ $GL_n \subseteq M_n$ = $n \times n$ matrices

open : $\det(x) \neq 0$

polynomial in the entries x_{ij}

Locus $\underline{\det(x) = 0}$ in M_n is closed (in Zariski topology, also in analytic topology)

x_{ij}, and new coordinate z

equation $\det(X) \cdot z - 1 = 0$ i.e., $z = \frac{1}{\det(X)}$.

$$X = \begin{pmatrix} x_{12}, \dots \\ \\ \end{pmatrix}$$

This Locus $\xrightarrow{\quad}$ M_n

$\hookrightarrow$, onto GL_n.

Y $\qquad$ (M_n)

$f \in \mathcal{O}(Y)$

function on Y

$Y \setminus (\underbrace{V(f)}_{\text{zero locus}})$ is open $\qquad$ $z = \frac{1}{f(y)}$

$Y \subseteq K^n$ is zero locus of polynomials $p(y_1, \dots, y_n)$

Affine variety in K^{n+1}

$y_1, \dots, y_n, z$

Old eq'ns $p(y) = 0$ of Y and $f(y) \cdot z - 1 = 0$ $\qquad$ $\neq 0$

Structure of Alg. Groups

$(\mathbb{C}, +)$ vs. $(\mathbb{C}^\times, \cdot)$

$K, +$ vs $K^\times, \cdot$

$\mathbb{G}_a(K)$ $\quad$ $\mathbb{G}_m(K) = GL_1$

$\mathcal{O}(\mathbb{G}_a) = K[x]$ $\quad$ $\mathcal{O}(\mathbb{G}_m) = K[t^{\pm 1}]$

$K[x,z]/(xz-1) = K[x^{\pm 1}]$

$K^2 = \mathbb{G}_a \times \mathbb{G}_a \xrightarrow{\mu} \mathbb{G}_a$

$(x,y) \longmapsto x+y$

$\mathbb{G}_m \times \mathbb{G}_m \xrightarrow{\mu} \mathbb{G}_m$

$(u,v) \longmapsto uv$

$\mu^\# f = f \circ \mu$

$\mathbb{R} \times \mathbb{R}$

$\mathbb{R} \times U_1$

$\mathcal{O}(\mathbb{G}_a \times \mathbb{G}_a) \xleftarrow{\mu^\#} \mathcal{O}(\mathbb{G}_a)$

$K[x,y] \longleftarrow K[z]$

$x+y \longmapsfrom z$

$\mathcal{O}(\mathbb{G}_m \times \mathbb{G}_m) \xleftarrow{\mu^\#} \mathcal{O}(\mathbb{G}_m)$

$K[u^{\pm 1}, v^{\pm 1}] \qquad K[t^{\pm 1}]$

$uv \longmapsfrom t$

As Lie gps$/\mathbb{C}$, both have Lie algebra $\mathbb{C}$

$\mathbb{G}_m$ is reductive : finite dim'l reps ($\mathbb{G}_m \to Gl_n$) are direct sums of irreducibles (irr. are 1-dim'l).

$\mathbb{G}_a$ is not

$x \mapsto \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

$\mathbb{G}_a \to Gl_2 \curvearrowright \mathbb{C}^2$ $\quad \mathbb{G}_a \; e_1 = e_1$ $\quad x \cdot e_2 = x e_1 + e_2$

$\mathbb{C}^2 / \mathbb{C} \cdot e_1 \cong \mathbb{C} e_2$ $\quad \equiv e_2 \mod \mathbb{C} e_1$

$\mathbb{G}_a$ acts trivially.

$\mathbb{G}_a$ is unipotent

if it were $\mathbb{C} \oplus \mathbb{C}$

$\Rightarrow \mathbb{G}_a$ acts trivially

General alg. group G / K

G is linear ($G \hookrightarrow Gl_n$) $\iff$ G is affine $G \hookrightarrow K^n$

finite group

Abelian variety.

G

$\cup$

G_0 . $\leftarrow$ identity component, connected.

$\cup$

G_{lin} $\leftarrow$ linear (= affine) normal subgroup

$\mathbb{G}_m$

Semisimple $\rightarrow$ $\cup$

(R = radical, solvable)

Reductive

$\cup$

R_u = unipotent radical

Gl_n

$\cup$ $\cup$

$\mathbb{C}^\times$ Sl_n

$\wr$

$\mathbb{C}^\times \cdot I$

$gl_n = \mathbb{C} \oplus sl_n$ Semisimple $\cup$

$\{e\}$

$\mathbb{C} \cdot I$ $\quad$ $tr = 0$

radical

Reductive linear alg. groups

Torus $= (\mathbb{G}_m)^n$